Int. J. Solids Structures Vol. 21, No. 1, pp. 67–77, 1985
Printed in the U.S.A.

0020–7683/85 $3.00+.00


# THE STRAIN ENERGY DENSITY OF RUBBER-LIKE SHELLS†,‡

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(*Received* 31 *October* 1983)

**Abstract**—Under the assumption that the state of a shell is described by the change in the first and second fundamental forms of its midsurface from an initially elastic isotropic state, an approximate strain energy density is derived, first using strictly two-dimensional arguments and then by descent from three dimensions assuming incompressibility. To within errors inherent in shell theory itself, it is shown that the strain energy density is the same as that of a plate of the same material.

## 1. INTRODUCTION

Shells that undergo large elastic strains are not uncommon: basketballs, bladders, red blood cells, or rubber hoses, for example. At a given point on the undeformed reference surface $S$ let $\gamma$, $\bar{\rho}$, and $b$ denote, respectively, measures of the magnitudes of the extensional and bending strains and the undeformed curvatures, and let $h$ be the undeformed thickness. In what follows we show that if $\gamma = O(1)$, then the exact strain energy $V$ per unit of area of $S$ may be approximated to within a relative error of $O(hb, h^2\bar{\rho}^2)$ by a function $\Phi$ *quadratic in the bending strains and of the same form as for a flat plate.* By definition of a shell, $hb$ is small: $h\bar{\rho}$ will be small except in the neighborhood of a fold (or near crease) where shell theory is not expected to hold anyway.

The paper has two parts. In the first, we determine $\Phi$ using intrinsic, two-dimensional arguments, based on work of Niordson [1] and Green & Naghdi [2]. However, these papers, unlike the present one, ultimately assume small extensional strains.

In the second part, we derive $\Phi$ from a three-dimensional strain energy density $W$. Here we follow Koiter [3], except that we do not assume that the extensional strains are small. In place of Koiter's assumption of plane stress we assume incompressibility. Either assumption serves to relate the transverse normal strain to the remaining strains. (The plane stress assumption is consistent with ours if Poisson's ratio $\nu = \frac{1}{2}$.)

A descent from three dimensions for rubber-like shells was pioneered by Biricikoglu & Kalnins [4] who took $W$ to be of the Mooney form. However, their analysis has two serious shortcomings. First, their constitutive law for the stress resultants contains an unknown hydrostatic pressure. While such a term appears in the three-dimensional stress–strain relations as a consequence of the incompressibility constraint, it should not occur in a shell (or membrane) theory because the midsurface can change area as it deforms. Second, the kinematic model adopted in [4] is too restrictive: it allows incompressibility to be satisfied only at the midsurface. Failure to enforce this constraint throughout the shell thickness leads to incorrect coefficients for the bending strains in $\Phi$.

For an infinite cylindrical shell, each cross section of which undergoes identical, planar deformation, our results, specialized to a Mooney material, agree with those obtained by Libai and me [5].

## 2. INTRINSIC ANALYSIS

The classical, nonlinear theory of shells assumes that the Internal Virtual Work, IVW, on an arbitrary, initial piece $P$ of the reference surface $S$ is given by [6, Eqn (53)]

† Dedication: To Eric Reissner on the occasion of his seventieth birthday in appreciation of his inspired teaching and his insights into shell theory.

‡ This work was supported by the National Science Foundation under Grant No. CEE-8117103.

$$IVW = \int_{\overline{P}} (\overline{n}^{\alpha\beta}\, \delta\gamma_{\alpha\beta} + \overline{m}^{\alpha\beta}\, \delta\overline{\rho}_{\alpha\beta})\, d\overline{A}\ , \tag{2.1}$$

where $\overline{P}$ is the image of $P$. In the integrand, $\overline{n}^{\alpha\beta}$ and $\overline{m}^{\alpha\beta}$ are stress resultants and couples and

$$\gamma_{\alpha\beta} = \frac{1}{2}(\overline{a}_{\alpha\beta} - a_{\alpha\beta}), \quad \overline{\rho}_{\alpha\beta} = \overline{b}_{\alpha\beta} - b_{\alpha\beta} \tag{2.2}$$

are extensional and bending strains. $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the covariant components of the metric and curvature tensors of $S$ and the barred quantities are their counterparts for the deformed reference surface $\overline{S}$.

A shell is said to be (isothermally) elastic if there exists a function $V$ of the strains—the strain energy per unit area of $S$—such that

$$IVW = \delta \int_P V\, dA = \int_P \delta V\, dA = \int_{\overline{P}} (a/\overline{a})^{1/2}\, \delta V\, d\overline{A}\ , \tag{2.3}$$

where $a = \det(a_{\alpha\beta})$, etc. As $P$ is arbitrary and as we may always construct a displacement field such that at any point $\delta\gamma_{\alpha\beta}$ and $\delta\overline{\rho}_{\alpha\beta}$ take on arbitrary values, it follows that

$$\overline{n}^{\alpha\beta} = \sqrt{\frac{a}{\overline{a}}}\frac{\partial V}{\partial\gamma_{\alpha\beta}}, \quad \overline{m}^{\alpha\beta} = \sqrt{\frac{a}{\overline{a}}}\frac{\partial V}{\partial\overline{\rho}_{\alpha\beta}}. \tag{2.4}$$

The equilibrium equations that $\overline{n}^{\alpha\beta}$ and $\overline{m}^{\alpha\beta}$ must satisfy are given by eqns (55) and (56) of [6].

To keep the analysis manageable, we assume that $V$ is anisotropic and inhomogeneous only through the curvature tensor of $S$. That is, we assume that at any point on $S$

$$V = \hat{V}(\Gamma, \overline{\mathbf{R}}, \mathbf{B}; h), \tag{2.5}$$

where

$$\Gamma = \gamma^{\alpha\beta}\, \mathbf{a}_\alpha \mathbf{a}_\beta, \quad \overline{\mathbf{R}} = \overline{\rho}^{\alpha\beta}\, \mathbf{a}_\alpha \mathbf{a}_\beta, \quad \mathbf{B} = b^{\alpha\beta}\, \mathbf{a}_\alpha \mathbf{a}_\beta. \tag{2.6}$$

In (2.6), $\mathbf{a}_\alpha = \partial\mathbf{p}/\partial\sigma^\alpha$ are the covariant base vectors associated with a set of Gaussian coordinates $(\sigma^\alpha)$ that determine the position $\mathbf{p}$ of a point on $S$ and $\mathbf{a}_\alpha\mathbf{a}_\beta$ denotes the direct (or dyadic) product of $\mathbf{a}_\alpha$ and $\mathbf{a}_\beta$.

The value of $V$ at a given point of $S$ must be independent of the coordinate system; thus $V$ depends only on the scalar invariants of the three, symmetric, second order tensors $\Gamma$, $\overline{\mathbf{R}}$, $\mathbf{B}$. Recalling the Cayley-Hamilton theorem for any two-dimensional, second order tensor $\mathbf{S}$,

$$\mathbf{S}^2 = (Tr\mathbf{S})\mathbf{S} + \frac{1}{2}(Tr\mathbf{S}^2 - Tr^2\mathbf{S})\mathbf{1}, \tag{2.7}$$

where $Tr$ denotes the trace and $\mathbf{1}$ is the identity tensor, we conclude that there are, at most, ten individual and combined invariants of $\Gamma$, $\overline{\mathbf{R}}$, and $\mathbf{B}$:

$$Tr\Gamma,\ Tr\Gamma^2,\ Tr\overline{\mathbf{R}},\ Tr\overline{\mathbf{R}}^2,\ Tr\mathbf{B},\ Tr\mathbf{B}^2 \tag{2.8}$$

$$Tr\Gamma\cdot\overline{\mathbf{R}},\ Tr\Gamma\cdot\mathbf{B},\ Tr\overline{\mathbf{R}}\cdot\mathbf{B} \tag{2.9}$$

$$Tr\Gamma\cdot\overline{\mathbf{R}}\cdot\mathbf{B}. \tag{2.10}$$

The last invariant is not independent of the first nine. By applying the Cayley-Hamilton theorem to $(\Gamma + \bar{\mathbf{R}})^2 - (\Gamma^2 + \bar{\mathbf{R}}^2)$, pre (or post) multiplying the right-side by $\mathbf{B}$, and noting that $\Gamma$, $\bar{\mathbf{R}}$, and $\mathbf{B}$ are symmetric, one can show that

$$2Tr\Gamma \cdot \bar{\mathbf{R}} \cdot \mathbf{B} = Tr\Gamma Tr\bar{\mathbf{R}} \cdot \mathbf{B} + Tr\bar{\mathbf{R}}Tr\mathbf{B} \cdot \Gamma + Tr\mathbf{B}Tr\Gamma \cdot \bar{\mathbf{R}} - Tr\Gamma Tr\bar{\mathbf{R}}Tr\mathbf{B}. \quad (2.11)$$

In fact, there are only eight functionally independent invariants, not nine. This is obvious if we note that at any point of $S$, we may always choose a coordinate system in which the off-diagonal components of one of our three tensors are zero. Thus the nine invariants can be expressed in terms of $2 + 3 + 3$ components. (The dependence of the nine invariants in (2.8) and (2.9) also emerges upon regarding two-dimensional, second order, symmetric tensors as elements of a three-dimensional vector space with $Tr\mathbf{S} \cdot \mathbf{T}$ as the inner product.) In our analysis it does no harm to keep all nine of the invariants: it turns out that $\Phi$ involves only the first seven.

As $V$ has units of [FORCE/LENGTH] and must vanish with $h$, and as $Tr\bar{\mathbf{R}}$ and $Tr\mathbf{B}$ have units of [1/LENGTH], $V$ can be written in the form

$$V = \tilde{E}h\hat{v}(\Gamma, h\bar{\mathbf{R}}, h\mathbf{B}), \quad (2.12)$$

where the material constant $\tilde{E}$ may be called Young's modulus. Also, $V$ must not change if the unit normal $\mathbf{n}$ to $S$ is replaced by $-\mathbf{n}$ and its deformed image $\bar{\mathbf{n}}$ is replaced by $-\bar{\mathbf{n}}$. That is, $V$ must not change if $\bar{\mathbf{R}}$ and $\mathbf{B}$ are simultaneously replaced by $-\bar{\mathbf{R}}$ and $-\mathbf{B}$. Thus $v$ must be a function of the form

$$v = f[Tr\Gamma, Tr\Gamma^2, h^2(\mathrm{I}, \mathrm{II}, \ldots, \mathrm{XIII})], \quad (2.13)$$

where

$$\mathrm{I} = Tr^2\bar{\mathbf{R}}, \quad \mathrm{II} = Tr\bar{\mathbf{R}}^2, \quad \mathrm{III} = Tr\bar{\mathbf{R}}Tr\Gamma \cdot \bar{\mathbf{R}}, \quad \mathrm{IV} = Tr^2\Gamma \cdot \bar{\mathbf{R}}$$

$$\mathrm{V} = Tr\Gamma \cdot \bar{\mathbf{R}}Tr\Gamma \cdot \mathbf{B}, \quad \mathrm{VI} = Tr\bar{\mathbf{R}}Tr\Gamma \cdot \mathbf{B}, \quad \mathrm{VII} = Tr\mathbf{B}Tr\Gamma \cdot \bar{\mathbf{R}} \quad (2.14)$$

$$\mathrm{VIII} = Tr^2\Gamma \cdot \mathbf{B}, \quad \mathrm{IX} = Tr\mathbf{B}Tr\Gamma \cdot \mathbf{B}, \quad \mathrm{X} = Tr\bar{\mathbf{R}}Tr\mathbf{B}$$

$$\mathrm{XI} = Tr\bar{\mathbf{R}} \cdot \mathbf{B}, \quad \mathrm{XII} = Tr^2\mathbf{B}, \quad \mathrm{XIII} = Tr\mathbf{B}^2. \quad (2.15)$$

Now take $\gamma$, $\bar{\rho}$ and $b$ as norms defined by

$$\gamma \equiv \| \Gamma \| = \sqrt{Tr\Gamma^2}, \quad \bar{\rho} = \| \bar{\mathbf{R}} \|, \quad b = \| \mathbf{B} \|. \quad (2.16)$$

With the aid of the Schwarz inequality for symmetric tensors,

$$Tr^2\mathbf{S} \cdot \mathbf{T} \leq Tr\mathbf{S}^2 Tr\mathbf{T}^2, \quad (2.17)$$

we have

$$\mathrm{I} \leq 2\bar{\rho}^2, \quad \mathrm{II} = \bar{\rho}^2, \quad |\mathrm{III}| \leq \sqrt{2}\gamma\bar{\rho}^2, \quad \mathrm{IV} \leq \gamma^2\bar{\rho}^2 \quad (2.18)$$

$$|\mathrm{V}| \leq \gamma^2 b\bar{\rho} \quad (2.19)$$

$$|\mathrm{VI}, \mathrm{VII}| \leq \sqrt{2}\gamma b\bar{\rho}, \quad \mathrm{VIII} \leq \gamma^2 b^2 \quad (2.20)$$

$$|\mathrm{IX}| \leq \sqrt{2}\gamma b^2, \quad \left| \frac{1}{2}\mathrm{X}, \mathrm{XI} \right| \leq b\bar{\rho} \quad (2.21)$$

$$\frac{1}{2}\mathrm{XII}, \mathrm{XIII} \leq b^2. \quad (2.22)$$

A further restriction we place on $f$ is that it be positive definite in the strains and that as $(\gamma, \overline{\rho}) \to 0$, (2.4) yield *linear* homogeneous stress–strain relations with no initial stress terms. This is guaranteed if we assume the existence of positive constants $\underline{A}$, $\underline{B}$, $\overline{A}$, $\overline{B}$ such that

$$\underline{A}\gamma^2 + \underline{B}h^2\overline{\rho}^2 \leq f \leq \overline{A}\gamma^2 + \overline{B}h^2\overline{\rho}^2, \quad \forall\gamma \leq \overline{\gamma},\ h\overline{\rho},\ hb \leq 1, \tag{2.23}$$

where $\overline{\gamma}$ is an O(1) constant.

We now assume that $f$ has a Taylor expansion in $h^2$ of the form

$$f = \varphi + \psi + \theta, \tag{2.24}$$

where

$$\varphi = A + h^2 B \tag{2.25}$$

$$B = B_1\mathrm{I} + B_2\mathrm{II} + B_3\mathrm{III} + B_4\mathrm{IV} \tag{2.26}$$

$$\psi = h^2(B_5 V + \cdots + B_{13}\mathrm{XIII}) \tag{2.27}$$

$$\theta = h^4(C_1\mathrm{I}^2 + C_2\mathrm{I}\cdot\mathrm{II} + \cdots + C_{91}\mathrm{XIII}^2), \tag{2.28}$$

and $A$, $B_1$, and $C_1$ are functions of $Tr\Gamma$ and $Tr\Gamma^2$. Note that (2.23) implies that

$$\underline{A}\gamma^2 \leq A \leq \overline{A}\gamma^2. \tag{2.29}$$

We are going to take $\varphi$ as our approximation to $f$. To exclude material instabilities in bending, we further assume that

$$\underline{B}\overline{\rho}^2 \leq B \leq \overline{B}\overline{\rho}^2. \tag{2.30}$$

To see what restrictions this places on the functions $B_1, \ldots, B_4$, let

$$\lambda_m = \min B, \quad \mathrm{I, II, III, IV\ fixed}, \quad \forall\overline{\rho} = 1. \tag{2.31}$$

In the Appendix it is shown that necessary and sufficient conditions for $\lambda_m$ to be positive, as (2.30) demands, are:

$$B_2 > 0 \tag{2.32}$$

$$2(B_1 + B_2) + B_3 Tr\Gamma + B_4 Tr\Gamma^2$$
$$> \{(2Tr\Gamma^2 - Tr^2\Gamma)(B_3 + B_4 Tr\Gamma)^2 + [2B_1 + B_3 Tr\Gamma + B_4(Tr^2\Gamma - Tr\Gamma^2)]\}^{1/2}. \tag{2.33}$$

We now obtain an upper bound on $\psi$ in terms of $\varphi$. To do so, we first note that (2.23) requires that $B_5, \ldots, B_{13}$ satisfy inequalities of the form

$$|B_5\gamma| \leq \overline{B}_5 \tag{2.34}$$

$$|\dot{B}_i| \leq \overline{B}_i, \quad i = 6, 7, 8 \tag{2.35}$$

$$|B_i| \leq \overline{B}_i\gamma, \quad i = 9, 10, 11 \tag{2.36}$$

$$|B_i| \leq \overline{B}_i\gamma^2, \quad i = 12, 13, \tag{2.37}$$

where the $\overline{B}_i$'s are positive constants. Let

$$\mu = \max(\underline{A}^{-1}, \underline{B}^{-1}). \tag{2.37}$$

Then with the aid of (2.19) to (2.22), (2.34) to (2.37), and Cauchy's inequality, we have

$$h^2 | B_5 V | \leq \overline{B}_5 h^2 \gamma b \overline{\rho} \leq \frac{1}{2} \overline{B}_5 h b (\gamma^2 + h^2 \overline{\rho}^2) \leq \frac{1}{2} \overline{B}_5 \mu (\underline{A} \gamma^2 + \underline{B} h^2 \overline{\rho}^2) \leq \frac{1}{2} \overline{B}_5 \mu h b \varphi \quad (2.38)$$

$$h^2 | B_5 \mathrm{VI}, B_6 \mathrm{VII} | \leq \sqrt{2} (\overline{B}_5, \overline{B}_6) h^2 \gamma b \overline{\rho} \leq \frac{1}{\sqrt{2}} (\overline{B}_5, \overline{B}_6) \mu h b \varphi \quad (2.39)$$

$$h^2 | B_8 \mathrm{VIII} | \leq \overline{B}_8 h^2 \gamma^2 b^2 \leq \overline{B}_8 h^2 b^2 (\gamma^2 + h^2 \overline{\rho}^2) \leq \overline{B}_8 \mu h^2 b^2 \varphi \quad (2.40)$$

$$h^2 | B_9 \mathrm{IX} | \leq \sqrt{2} \overline{B}_9 h^2 \gamma^2 b^2 \leq \sqrt{2} \overline{B}_9 \mu h^2 b^2 \varphi \quad (2.41)$$

$$h^2 | B_{10} \mathrm{X}, B_{11} \mathrm{XI} | \leq 2 (\overline{B}_{10}, \overline{B}_{11}) h^2 \gamma b \overline{\rho} \leq (\overline{B}_{10}, \overline{B}_{11}) \mu h b \varphi \quad (2.42)$$

$$h^2 | B_{12} \mathrm{XII}, B_{13} \mathrm{XIII} | \leq 2 (\overline{B}_{12}, \overline{B}_{13}) h^2 \gamma^2 b^2 \leq 2 (\overline{B}_{12}, \overline{B}_{13}) \mu h^2 b^2 \varphi. \quad (2.43)$$

Equations (2.38) to (2.43) show that $\psi = O(hb\varphi)$.

To bound $\theta$ in terms of $\varphi$, we assume that $C_1$ to $C_{88}$ are bounded while the coefficients of $\mathrm{XII}^2$, $\mathrm{XII} \cdot \mathrm{XIII}$, and $\mathrm{XIII}^2$ satisfy

$$| C_i | \leq \overline{C}_1 \gamma^2, \quad i = 89, 90, 91, \quad (2.44)$$

where the $\overline{C}_i$s are constants. The dominant terms in $\theta$ are those involving I and II only and XII and XIII only. From (2.18), (2.22), and (2.44) we have

$$\begin{aligned} h^4 | C_1 \mathrm{I}^2, C_2 \mathrm{I} \cdot \mathrm{II}, C_{14} II^2 | &\leq 2 (\overline{C}_1, \overline{C}_2, \overline{C}_{14}) h^4 \overline{\rho}^4 \\ &\leq 2 (\overline{C}_1, \overline{C}_2, \overline{C}_{14}) h^2 \overline{\rho}^2 (\gamma^2 + h^2 \overline{\rho}^2) \\ &\leq 2 (\overline{C}_1, \overline{C}_2, \overline{C}_{14}) \mu h^2 \overline{\rho}^2 \varphi \quad (2.45) \end{aligned}$$

$$\begin{aligned} h^4 | C_{89} \mathrm{XII}^2, C_{90} \mathrm{XII} \cdot \mathrm{XIII}, C_{91} \mathrm{XIII}^2 | &\leq (\overline{C}_{89}, \overline{C}_{90}, \overline{C}_{91}) h^4 \gamma^2 b^4 \\ &\leq (\overline{C}_{89}, \overline{C}_{90}, \overline{C}_{91}) \mu h^4 b^4 \varphi. \quad (2.46) \end{aligned}$$

Thus $\theta = O[(hb, h^2 \overline{\rho}^2) \varphi]$.

In summary, we have shown that $V = \Phi[1 + O(hb, h^2 \overline{\rho}^2)]$, where

$$\Phi = \tilde{E} h \{ A + h^2 [ B_1 (\overline{\rho}^\alpha_\alpha)^2 + B_2 \overline{\rho}^\alpha_\beta \overline{\rho}^\beta_\alpha + B_3 \gamma^\alpha_\beta \overline{\rho}^\beta_\alpha \overline{\rho}^\lambda_\lambda + B_4 (\gamma^\alpha_\beta \overline{\rho}^\beta_\alpha)^2 ] \}, \quad (2.47)$$

and $A, B_1, \ldots, B_4$ are functions of $\gamma^\alpha_\alpha$ and $\gamma^\alpha_\beta \gamma^\beta_\alpha$ that must satisfy (2.29), (2.32) and (2.33). As the curvature tensor of $S$ does not appear in (2.47), $\Phi$ for a shell is the same as for a plate of the same material.

A major simplification occurs if we assume that the $B_i$s in (2.47) have Taylor expansions of the form

$$B_i = B_i^0 + B_i^1 \gamma^\lambda_\lambda + O(\gamma^2), \quad (2.48)$$

where $B_i^0$ and $B_i^1$ are constants. Then, to within the error made in replacing $V$ by $\Phi$, we may replace (2.47) by

$$\Phi = \tilde{E} h \{ A + h^2 [ (B_1^0 + B_1^1 \gamma^\lambda_\lambda)(\overline{\rho}^\alpha_\alpha)^2 + (B_2^0 + B_2^1 \gamma^\lambda_\lambda) \overline{\rho}^\alpha_\beta \overline{\rho}^\beta_\alpha + B_3^0 \gamma^\alpha_\beta \overline{\rho}^\beta_\alpha \overline{\rho}^\lambda_\lambda ] \}. \quad (2.49)$$

Definitions of the bending strains other than $(2.2)_2$ are often convenient. One, suggested by the work of the next section, is

$$\hat{\rho}_{\alpha\beta} \equiv \sqrt{a/\overline{a}}\, \overline{b}_{\alpha\beta} - b_{\alpha\beta}. \quad (2.50)$$

As

$$\overline{\rho}_{\alpha\beta} = \sqrt{\overline{a}/a} (\hat{\rho}_{\alpha\beta} + b_{\alpha\beta}) - b_{\alpha\beta} \quad (2.51)$$

and

$$\bar{a}/a = det(\delta^{\alpha}_{\beta} + 2\gamma^{\alpha}_{\beta}) = 1 + 2\gamma^{\alpha}_{\alpha} + 2(\gamma^{\alpha}_{\alpha}\gamma^{\beta}_{\beta} - \gamma^{\alpha}_{\beta}\gamma^{\beta}_{\alpha}), \tag{2.52}$$

it follows that

$$\delta\bar{\rho}_{\alpha\beta} = \sqrt{\bar{a}/a}\,[\delta\hat{\rho}_{\alpha\beta} + (\hat{\rho}_{\alpha\beta} + b_{\alpha\beta})\bar{\bar{a}}^{\lambda\mu}\delta\gamma_{\lambda\mu}, \tag{2.53}$$

where

$$\bar{\bar{a}}^{\lambda\mu} = (a/\bar{a})[(1 + 2\gamma^{\alpha}_{\alpha})a^{\lambda\mu} - 2\gamma^{\lambda\mu}], \quad \bar{\bar{a}}^{\lambda\mu}\bar{a}_{\mu\nu} = \delta^{\lambda}_{\nu}. \tag{2.54}$$

Introducing the modified stress resultants and couples

$$n^{\alpha\beta} = \sqrt{\bar{a}/a}\,\bar{n}^{\alpha\beta} + \bar{\bar{a}}^{\alpha\beta}(\hat{\rho}_{\lambda\mu} + b_{\lambda\mu})m^{\lambda\mu}, \quad m^{\alpha\beta} = (\bar{a}/a)\bar{m}^{\alpha\beta}, \tag{2.55}$$

we have

$$IVW = \int_P (n^{\alpha\beta}\,\delta\gamma_{\alpha\beta} + m^{\alpha\beta}\,\delta\hat{\rho}_{\alpha\beta})\,dA = \int_P \delta\mathrm{V}\,dA. \tag{2.56}$$

Hence,

$$n^{\alpha\beta} = \frac{\partial \mathrm{V}}{\partial\gamma_{\alpha\beta}}, \quad m^{\alpha\beta} = \frac{\partial \mathrm{V}}{\partial\hat{\rho}_{\alpha\beta}}. \tag{2.57}$$

The strain energy density V can again be approximated by a function of the form (2.47) or (2.49), save that the $B$s must be everywhere replaced by new $\hat{B}$s.

## 3. DESCENT FROM THREE DIMENSIONS

The intrinsic approach has shown that a sufficiently smooth two-dimensional strain energy density for a materially isotropic shell depends, to a first approximation, on five functions of the two extensional strain invariants. These functions must be either determined experimentally, or else, as follows, wrung from an assumed three-dimensional strain energy density $W$. By the results of the preceding section, we need look only at plates. This observation simplifies things considerably.

Let the undeformed plate be materially isotropic. That is, assume that $W = W(\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3)$, where [7]

$$\mathrm{I}_1 = 3 + 2E^i_i, \quad \mathrm{I}_2 = 3 + 4E^i_i + 2(E^i_iE^j_j - E^i_jE^j_i), \quad \mathrm{I}_3 = det(\delta^i_j + 2E^i_j) \tag{3.1}$$

are the scalar invariants of $\mathbf{1} + 2\mathbf{E}$ and

$$\mathbf{E} = \frac{1}{2}(\bar{\mathbf{g}}_i \cdot \bar{\mathbf{g}}_j - \mathbf{g}_i \cdot \mathbf{g}_j)\mathbf{g}^i\mathbf{g}^j$$
$$\equiv E_{ij}\mathbf{g}^i\mathbf{g}^j \equiv E^i_j\mathbf{g}_i\mathbf{g}^j \tag{3.2}$$

is the three-dimensional Lagrangian strain tensor. In (3.2), $\mathbf{g}_i = \partial\mathbf{x}/\partial\sigma^i$ are the covariant base vectors of a referential coordinate system that locates the position $\mathbf{x}$ of a particle in the undeformed body and $\bar{\mathbf{g}}_i = \partial\bar{\mathbf{x}}/\partial\sigma^i$, where $\bar{\mathbf{x}}$ is the position of the same particle after deformation; $\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j$.

Specializing to plate coordinates, we have

$$\mathbf{x} = \mathbf{p}(\sigma^{\alpha}) + \zeta\mathbf{k}, \quad |\zeta| \le \frac{1}{2}h(\sigma^{\alpha}). \tag{3.3}$$

As the notation suggests, $\sigma^3 = \zeta$, $S$ has been taken as the undeformed midplane, and **k** is a fixed unit vector perpendicular to $S$. Thus

$$\mathbf{g}_\alpha = \mathbf{a}_\alpha, \quad \mathbf{g}_3 = \mathbf{k} \tag{3.4}$$

and the invariants in (3.1) take the expanded form

$$I_1 = 2(1 + T) + 1 + 2E \tag{3.5}$$

$$I_2 = J + 2(1 + T)(1 + 2E) - 4E^\alpha E_\alpha \tag{3.6}$$

$$I_3 = J(1 + 2E) + 8E^\alpha_\beta E_\alpha E^\beta - 4(1 + 2T)E^\alpha E_\alpha, \tag{3.7}$$

where

$$T = E^\alpha_\alpha, \quad D = \frac{1}{2}(E^\alpha_\alpha E^\beta_\beta - E^\alpha_\beta E^\beta_\alpha), \quad J = 1 + 2T + 4D, \tag{3.8}$$

$T$ and $D$ being mnemonics for "trace" and "determinant."

In rubber-like bodies it is usually assumed that $W = W(I_1, I_2)$ and that $I_3 = 1$ (incompressibility). From (3.7), this last condition implies that

$$1 + 2E = [1 + 4(1 + 2T)E^\alpha E_\alpha - 8E^\alpha_\beta E_\alpha E^\beta]J^{-1}. \tag{3.9}$$

Let $\sqrt{a}\, d\sigma^1\, d\sigma^2\, d\zeta \equiv dA\, d\zeta$ denote the differential element of volume in the undeformed plate. Then the strain energy per unit area of $S$ is given by

$$\begin{aligned} V &= \int_{-h/2}^{h/2} W(I_1, I_2)\, d\zeta \\ &= \frac{1}{2}\bar{E}h \int_1^1 w(I_1, I_2)\, dz, \end{aligned} \tag{3.10}$$

where

$$W = \bar{E}w, \quad \zeta = \frac{1}{2}hz. \tag{3.11}$$

To evaluate the integral we need the explicit $z$ dependence of $E_{\alpha\beta}$ and $E_\alpha$. We adopt the simplest kinematic model of displacement that permits significant normal strains, negligible transverse shearing strains, and exact satisfaction of incompressibility, namely

$$\bar{\mathbf{x}} = \bar{\mathbf{p}}(\sigma^\alpha) + g(\sigma^\alpha, \zeta)\bar{\mathbf{n}}(\sigma^\alpha). \tag{3.12}$$

Then

$$\bar{\mathbf{g}}_\alpha = (\delta^\lambda_\alpha - g\bar{a}^{\lambda\mu}\bar{p}_{\mu\alpha})\bar{\mathbf{a}}_\lambda + g_{,\alpha}\bar{\mathbf{n}}, \quad \bar{\mathbf{g}}_3 = g_{,3}\bar{\mathbf{n}}, \tag{3.13}$$

where $g$ is to be chosen presently. Substituting (3.4) and (3.13) into (3.2), we get

$$E_{\alpha\beta} = \gamma_{\alpha\beta} - g\bar{\rho}_{\alpha\beta} + \frac{1}{2}(g^2C_{\alpha\beta} + g_{,\alpha}g_{,\beta}) \tag{3.14}$$

$$E_\alpha = \frac{1}{2}g_{,\alpha}g_{,3} \tag{3.15}$$

$$E = \frac{1}{2}(g^2_{,3} - 1), \tag{3.16}$$

where, with the aid of (2.51),

$$\begin{aligned} C^\alpha_\beta &= \bar{\bar{a}}^{\lambda\mu}\bar{\rho}^\alpha_\lambda\bar{\rho}_{\mu\beta} \\ &= (1 + 2\gamma^\lambda_\lambda)\hat{\rho}^\alpha_\mu\hat{\rho}^\mu_\beta - 2\hat{\rho}^\alpha_\lambda\gamma^\lambda_\mu\hat{\rho}^\mu_\beta \\ &= \left(\frac{1}{2}\delta^\alpha_\beta + \gamma^\alpha_\beta\right)[\hat{\rho}^\lambda_\mu\hat{\rho}^\mu_\lambda - (\hat{\rho}^\lambda_\lambda)^2] + [(1 + 2\gamma^\lambda_\lambda)\hat{\rho}^\mu_\mu - 2\gamma^\lambda_\mu\hat{\rho}^\mu_\lambda]\hat{\rho}^\alpha_\beta. \end{aligned} \tag{3.17}$$

The procedure for obtaining an approximate two-dimensional strain energy density $\Phi$ is now clear. We substitute (3.14) and (3.15) into (3.5) and (3.6) and the resulting expressions, along with $(3.11)_2$, into (3.10). As $w$ is a known function of $I_1$ and $I_2$, the integrand becomes a known function of $z$. Integrating, we obtain an expression of the form

$$\begin{aligned} V &= \bar{E}h[A + h^2B + O(h^4)] \\ &= \bar{E}h\left\{w_0 + \frac{h^2}{24}\left[\left[\frac{\partial w}{\partial I_1}\right]_0\left[\frac{d^2 I_1}{d\zeta^2}\right]_0 + \left[\frac{\partial^2 w}{\partial I_1^2}\right]_0\left[\frac{dI_1}{d\zeta}\right]_0^2 + \cdots\right] + O(h^4)\right\}, \end{aligned} \tag{3.18}$$

where $w_0 = w(I_1, I_2)$ at $\zeta = 0$, etc.

To determine $g$ set

$$g(\sigma^\alpha, \zeta) = \zeta[g_0(\sigma^\alpha) + \zeta g_1(\sigma^\alpha) + \cdots]. \tag{3.19}$$

We impose incompressibility on our kinematic model by substituting (3.14) and (3.15), with $g$ given by (3.19), into (3.9). Equating coefficients of like powers of $\zeta$, we obtain an infinite sequence of algebraic equations. The solutions of the first two are

$$g_0 = \sqrt{a/\bar{a}}, \quad g_1/g_0 = \frac{1}{2}(a/\bar{a})[(1 + 2\gamma^\alpha_\alpha)\hat{\rho}^\beta_\beta - 2\gamma^\alpha_\beta\hat{\rho}^\beta_\alpha]. \tag{3.20}$$

To obtain formulas for $A$ and the $\hat{B}$s, let $T = T_0 + \zeta T_1 + \cdots$ and $J = J_0 + \zeta J_1 + \cdots$. It then follows from (2.50), (3.8), (3.14), and (3.19) that

$$T_0 = \gamma^\alpha_\alpha = Tr\mathbf{\Gamma}, \quad T_1 = -\hat{\rho}^\alpha_\alpha = -Tr\hat{\mathbf{R}} \tag{3.21, 3.22}$$

$$\begin{aligned} T_2 &= \frac{1}{2}(C^\alpha_\alpha + a^{\alpha\beta}g_{0,\alpha}g_{0,\beta}) - (g_1/g_0)\hat{\rho}^\alpha_\alpha \\ &= \frac{1}{2}(a/\bar{a})(1 + Tr\mathbf{\Gamma})(\hat{I}\hat{I} - \hat{I}) + O(\gamma^2/L^2) \end{aligned} \tag{3.23}$$

$$J_0 = \bar{a}/a \tag{3.24}$$

$$J_1 = -2[(1 + 2\gamma^\alpha_\alpha)\hat{\rho}^\beta_\beta - 2\gamma^\alpha_\beta\hat{\rho}^\beta_\alpha] = -2[(1 + 2Tr\mathbf{\Gamma})Tr\hat{\mathbf{R}} - 2Tr\mathbf{\Gamma}\cdot\hat{\mathbf{R}}] \tag{3.25}$$

$$J_2 = \hat{I} - \hat{I}\hat{I} + O(\gamma^2/L^2), \tag{3.26}$$

where, with reference to (2.14), $\hat{I} = (\hat{\rho}^\alpha_\alpha)^2$, etc. and $L$ is "the wavelength of the deformation pattern" [3] defined here, somewhat arbitrarily, as

$$L^2 = \gamma^2/a^{\alpha\beta}g_{0,\alpha}g_{0,\beta}. \tag{3.27}$$

($L$ could be 0 or $\infty$.) Substituting the expansions for $T$ and $J$ along with (3.9), (3.14), and (3.15) into (3.5) and (3.6), we have

$$
\begin{aligned}
\mathrm{I}_1 = {} & 2 + a/\bar{a} + 2Tr\boldsymbol{\Gamma} \\
& - 2\zeta\{[1 - (a/\bar{a})^2(1 + 2Tr\boldsymbol{\Gamma})]Tr\hat{\mathbf{R}} + 2(a/\bar{a})^2 Tr\boldsymbol{\Gamma}\cdot\hat{\mathbf{R}}]\} \\
& + \zeta^2\{(a/\bar{a})[4(a/\bar{a})^2(1 + 2Tr\boldsymbol{\Gamma})^2 - 1 - a/\bar{a} - Tr\boldsymbol{\Gamma}]\hat{\mathrm{I}} \\
& + (a/\bar{a})(1 + a/\bar{a} + Tr\boldsymbol{\Gamma})\hat{\mathrm{I}}\hat{\mathrm{I}} \\
& - 16(a/\bar{a})^3(1 + 2Tr\boldsymbol{\Gamma})\mathrm{I}\hat{\mathrm{I}}\mathrm{I} + 16(a/\bar{a})^3\mathrm{I}\hat{\mathrm{V}} + \mathrm{O}(\gamma^2/L^2)\} + \cdots
\end{aligned} \tag{3.28}
$$

$$
\begin{aligned}
\mathrm{I}_2 = {} & \bar{a}/a + 2(a/\bar{a})(1 + Tr\boldsymbol{\Gamma}) \\
& - 2\zeta\{[1 + a/\bar{a} + 2Tr\boldsymbol{\Gamma} - 2(a/\bar{a})^2(1 + Tr\boldsymbol{\Gamma})(1 + 2Tr\boldsymbol{\Gamma})]Tr\hat{\mathbf{R}} \\
& - 2[1 - 2(a/\bar{a})^2(1 + Tr\boldsymbol{\Gamma})]Tr\boldsymbol{\Gamma}\cdot\hat{\mathbf{R}}\} \\
& + \zeta^2\{[1 - (a/\bar{a})^2(7 + 11Tr\boldsymbol{\Gamma}) + 8(a/\bar{a})^3(1 + Tr\boldsymbol{\Gamma})(1 + 2Tr\boldsymbol{\Gamma})^2]\hat{\mathrm{I}} \\
& + [-1 + 3(a/\bar{a})^2(1 + Tr\boldsymbol{\Gamma})]\hat{\mathrm{I}}\hat{\mathrm{I}} + 8(a/\bar{a})^2[1 - 4(a/\bar{a})(1 + Tr\boldsymbol{\Gamma})(1 + 2Tr\boldsymbol{\Gamma})]\mathrm{I}\hat{\mathrm{I}}\mathrm{I} \\
& + 32(a/\bar{a})^3(1 + Tr\boldsymbol{\Gamma})\mathrm{I}\hat{\mathrm{V}} + \mathrm{O}(\gamma^2/L^2)\} + \cdots
\end{aligned} \tag{3.29}
$$

Substituting (3.28) and (3.29) into (3.18) and comparing the resulting expression with (2.47) (with $B$s replaced by $\hat{B}$s), we obtain

$$
A = w_0 \tag{3.30}
$$

$$
\begin{aligned}
\hat{B}_1 = {} & \frac{1}{12}\left\{\left[\frac{\partial w}{\partial \mathrm{I}_1}\right]_0 (a/\bar{a})[4(a/\bar{a})^2(1 + 2Tr\boldsymbol{\Gamma})^2 - 1 - a/\bar{a} - Tr\boldsymbol{\Gamma}]\right. \\
& \left. + \left[\frac{\partial w}{\partial \mathrm{I}_2}\right]_0 [1 - (a/\bar{a})^2(7 + 11Tr\boldsymbol{\Gamma}) + 8(a/\bar{a})^3(1 + Tr\boldsymbol{\Gamma})(1 + 2Tr\boldsymbol{\Gamma})^2]\right\} \\
& + \frac{1}{6}\left\{\left[\frac{\partial^2 w}{\partial \mathrm{I}_1^2}\right]_0 [1 - (a/\bar{a})^2(1 + 2Tr\boldsymbol{\Gamma})]^2\right. \\
& \left. + \left[\frac{\partial^2 w}{\partial \mathrm{I}_2^2}\right]_0 [1 + a/\bar{a} + 2Tr\boldsymbol{\Gamma} - 2(a/\bar{a})^2(1 + Tr\boldsymbol{\Gamma})(1 + 2Tr\boldsymbol{\Gamma})]^2\right\} \\
= {} & \frac{1}{12}\left\{\left[\frac{\partial w}{\partial \mathrm{I}_1}\right]_0 (2 - 3Tr\boldsymbol{\Gamma}) + \left[\frac{\partial w}{\partial \mathrm{I}_2}\right]_0 (2 + 9Tr\boldsymbol{\Gamma})\right\} + \mathrm{O}(\gamma^2)
\end{aligned} \tag{3.31}
$$

$$
\begin{aligned}
\hat{B}_2 = {} & \frac{1}{12}\left\{\left[\frac{\partial w}{\partial \mathrm{I}_1}\right]_0 (a/\bar{a})(1 + a/\bar{a} + Tr\boldsymbol{\Gamma}) + \left[\frac{\partial w}{\partial \mathrm{I}_2}\right]_0 [-1 + 3(a/\bar{a})^2(1 + Tr\boldsymbol{\Gamma})]\right\} \\
= {} & \frac{1}{12}\left\{\left[\frac{\partial w}{\partial \mathrm{I}_1}\right]_0 (2 - 5Tr\boldsymbol{\Gamma}) + \left[\frac{\partial w}{\partial \mathrm{I}_2}\right]_0 (2 - 9Tr\boldsymbol{\Gamma}) + \right\} + \mathrm{O}(\gamma^2)
\end{aligned} \tag{3.32}
$$

$$
\begin{aligned}
\hat{B}_3 = {} & \frac{2}{3}\left\{-2\left[\frac{\partial w}{\partial \mathrm{I}_1}\right]_0 (a/\bar{a})^3(1 + 2Tr\boldsymbol{\Gamma})\right. \\
& + \left[\frac{\partial w}{\partial \mathrm{I}_2}\right]_0 (a/\bar{a})^2[1 - 4(a/\bar{a})(1 + Tr\boldsymbol{\Gamma})(1 + 2Tr\boldsymbol{\Gamma})] \\
& + \left[\frac{\partial^2 w}{\partial \mathrm{I}_1^2}\right]_0 (a/\bar{a})^2[1 - (a/\bar{a})^2(1 + 2Tr\boldsymbol{\Gamma})]
\end{aligned}
$$

$$- \left[\frac{\partial^2 w}{\partial I_2^2}\right]_0 [1 - 2(a/\bar{a})^2(1 + Tr\Gamma)][1 + a/\bar{a} + 2Tr\Gamma - 2(a/\bar{a})^2(1 + Tr\Gamma)(1 + 2Tr\Gamma)]\Bigg\}$$

$$= -\frac{4}{3}\left[\frac{\partial w}{\partial I_1}\right]_0 - 2\left[\frac{\partial w}{\partial I_2}\right]_0 + O(\gamma) \tag{3.33}$$

$$\hat{B}_4 = \frac{2}{3}\Bigg\{2\left[\frac{\partial w}{\partial I_1}\right]_0 (a/\bar{a})^3 + 4\left[\frac{\partial w}{\partial I_2}\right]_0 (a/\bar{a})^3 (1 + Tr\Gamma)$$
$$+ \left[\frac{\partial^2 w}{\partial I_1^2}\right]_0 (a/\bar{a})^4 + \left[\frac{\partial^2 w}{\partial I_2^2}\right]_0 [1 - 2(a/\bar{a})^2 (1 + Tr\Gamma)]^2\Bigg\}$$
$$= O(1). \tag{3.34}$$

## 4. CYLINDRICAL BENDING OF PLATES

If a plate is bent into a right cylindrical shell, $E_2^1 = E_2^2 = 0$ and hence

$$Tr\Gamma = \gamma = \gamma_1^1, \quad \gamma_2^1 = \gamma_2^2 = 0$$
$$Tr\hat{\mathbf{R}} = \hat{\rho} = \hat{\rho}_1^1, \quad \hat{\rho}_2^1 = \hat{\rho}_2^2 = 0 \tag{4.1}$$
$$\hat{I} = \hat{II} = \hat{\rho}^2, \quad \hat{III} = \gamma\hat{\rho}^2, \quad \hat{IV} = \gamma^2\hat{\rho}^2.$$

It follows from (3.5) to (3.8) that

$$J \equiv I_1 = I_2 = 2(1 + E_1^1) + \frac{1 + 4(E_1^3)^2}{1 + 2E_1^1}$$
$$= 2(1 + \gamma) + \frac{1}{1 + 2\gamma} - \zeta\left[\frac{8\gamma(1 + \gamma)}{(1 + 2\gamma)^2}\hat{\rho}\right]$$
$$+ \zeta^2\left[\frac{4\hat{\rho}^2}{(1 + 2\gamma)^3} + O(\gamma^2/L^2)\right] + \cdots \tag{4.2}$$

Substituting (4.2) into (3.18) and recalling that $\Phi = \bar{E}h\varphi$, we find that

$$\varphi = w(J_0, J_0) + \frac{h^3}{3}\Bigg\{\left[\left[\frac{\partial w}{\partial I_1}\right]_0 + \left[\frac{\partial w}{\partial I_2}\right]_0\right]\frac{1}{(1 + 2\gamma)^3}$$
$$+ 8\left[\left[\frac{\partial^2 w}{\partial I_1^2}\right]_0 + \left[\frac{\partial^2 w}{\partial I_2^2}\right]_0\right]\frac{\gamma^2(1 + \gamma)^2}{(1 + 2\gamma)^4}\Bigg\}\hat{\rho}^2 + O(h^2\gamma^2/L^2). \tag{4.3}$$

For a Mooney material

$$W = C_1(I_1 - 3) + C_2(I_2 - 3)$$
$$\equiv \bar{E}(J - 3), \tag{4.4}$$

where $C_1$ and $C_2$ are constants.

If we set

$$\lambda^2 = 1 + 2\gamma, \quad \lambda\kappa' \equiv \lambda\bar{b}_1^1 = \lambda\bar{a}^{11}\bar{\rho}_{11} = \hat{\rho}_{11} = \hat{\rho} \tag{4.5}$$

(i.e. if we introduce the physical bending strain $\kappa'$ which is just the curvature of the deformed, cylindrical midsurface), then (4.3) reduces to

$$\varphi = \lambda^2 + \frac{1}{\lambda^2} - 2 + \frac{h^2}{3}\left[\frac{\kappa'^2}{\lambda^4}\right] + O(h^2\gamma^2/L^2, h^2\bar{\rho}^2). \tag{4.6}$$

This equation agrees with equation (15c) of [5] provided that, in the latter, we set $\kappa\mu = \lambda\kappa'$ and then let $\kappa$, the undeformed curvature, approach zero.

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## APPENDIX

We may always choose a coordinate system on $S$ such that, at any given point $p$, $\{a_\alpha, n\}$ is orthonormal and $\gamma_{12} = 0$. At $p$, all covariant components are physical components and we have (dropping overbars)

$$\begin{aligned} B &= B_1(\rho_{11} + \rho_{22})^2 + B_2(\rho_{11}^2 + \rho_{22}^2 + 2\rho_{12}^2) \\ &\quad + B_3(\rho_{11} + \rho_{22})(\gamma_{11}\rho_{11} + \gamma_{22}\rho_{22}) + B_4(\gamma_{11}\rho_{11} + \gamma_{22}\rho_{22})^2 \\ &= (B_1 + B_2 + B_3\gamma_{11} + B_4\gamma_{11}^2)\rho_{11}^2 + 2(B_1 + \tfrac{1}{2}B_3 Tr\Gamma + B_4\gamma_{11}\gamma_{22})\rho_{11}\rho_{22} \\ &\quad + (B_1 + B_2 + B_3\gamma_{22} + B_4\gamma_{22}^2)\rho_{22}^2 + 2B_2\rho_{12}^2 \\ &\equiv A_{11}\rho_{11}^2 + 2A_{12}\rho_{11}\rho_{22} + A_{22}\rho_{22}^2 + 2B_2\rho_{12}^2 > 0, \quad \forall\rho \neq 0. \end{aligned} \tag{A1}$$

The quadratic form $B$ is positive definite if and only if

$$A_{11} + A_{22} > 0, \quad A_{11}A_{22} > A_{12}^2, \quad B_2 > 0. \tag{A2}$$

Upon noting that

$$A_{11}A_{22} = \frac{1}{4}[(A_{11} + A_{22})^2 - (A_{11} - A_{22})^2], \tag{A3}$$

we may replace the first two conditions by the single requirement that

$$A_{11} + A_{22} > \sqrt{(A_{11} - A_{22})^2 + 4A_{12}^2} \tag{A4}$$

which, when the $A_{ij}$s are expressed in terms of the $B_i$s and the strain components, becomes (2.33).